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Stickelberger elements over rational function fields

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ABSTRACT

The family of Stickelberger elements associated to certain abelian extensions over global rational function fields is represented by a universal polynomial. This result is then used to prove a conjecture of Gross for these Stickelberger elements.

1. Introduction

In this paper we study the family of Stickelberger elements associated to certain abelian extensions over global rational function fields and apply our result to a conjecture of Gross.

Let K be a global function field and let S and T be two non-empty disjoint finite sets of places of K. Let L/K be a finite abelian extension unramified outside S with $\mathrm{Gal}(L/K) = G$. For each character $\chi \in \widehat{G}$, consider the modified L-function [6]

$$L(\chi, s) = \prod_{v \notin S} \bigl(1 - \chi(\mathrm{Fr}_v) N(v)^{-s}\bigr)^{-1} \times \prod_{v \in T} \bigl(1 - \chi(\mathrm{Fr}_v) N(v)^{1-s}\bigr) \quad \bigl(\mathrm{Re}(s) > 1\bigr), \tag{1}$$

where Fr_v is the Frobenius element at v and $N(v)$ is the norm.

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The Stickelberger element $\theta_{S,T,G}$, by definition [6], is the unique element of the group ring $\mathbb{C}[G]$ such that for every $\chi \in \widehat{G}$,

$$\chi(\theta_{S,T,G}) = L(\chi, 0).$$

Gross proves that $\theta_{S,T,G} \in \mathbb{Z}[G]$ in the function field case [6, Proposition 3.7]. He also proposes a conjecture about the residue class of $\theta_{S,T,G}$ modulo $I^{|S|}$. Here the augmentation ideal I is the kernel of the ring homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ sending $g \in G$ to 1. If it causes no confusion, we will use $\theta_{S,T}$ to denote $\theta_{S,T,G}$. The Stickelberger elements enjoy the following functorial property: if M/K is a sub-extension of L/K with $\mathrm{Gal}(M/K) = G'$, then the natural projection $G \to G'$ induces a ring homomorphism between the group rings, and under this homomorphism, $\theta_{S,T,G}$ is sent to $\theta_{S,T,G'}$. Thus for a profinite abelian extension L/K unramified outside S, one can define the Stickelberger element $\theta_{S,T,G}$ as the projective limit of the Stickelberger elements $\theta_{S,T,G'}$ where G' runs through all the finite quotients of G. In this case $\theta_{S,T,G}$ is in the (complete) group ring $\mathbb{Z}[G]$ which is the projective limit of the group rings $\mathbb{Z}[G']$, and the augmentation ideal $I = I_G$ is the projective limit of the augmentation ideals $I_{G'}$ of $\mathbb{Z}[G']$.

For the rest of this paper, we assume that we are in the case where K is the rational function field $\mathbb{F}_q(x)$, T is formed by a single place which is the zero of an irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ with $\deg(f) = d$ and S is of cardinality $n+1$, containing the pole, $\infty =: v_0$ of x, together with places $v_1, \ldots, v_n$ corresponding to linear polynomials $x - s_i$, $s_i \in \mathbb{F}_q$ for $i = 1, \ldots, n$. Also, we assume L/K is the maximal abelian extension unramified outside S and at worst tamely ramified on S. We will fix this setting while allowing the set of data $\{q, f, s_1, \ldots, s_n\}$ to vary. We shall call the tuple $(q, f, s_1, \ldots, s_n)$ a basic datum.

We have

$$L = \bar{\mathbb{F}}_q\big(\sqrt[q-1]{x - s_1},\ \sqrt[q-1]{x - s_2}, \ldots,\ \sqrt[q-1]{x - s_n}\big).$$

Put

$$L_\infty = \mathbb{F}_q\big(\sqrt[q-1]{x - s_1},\ \sqrt[q-1]{x - s_2}, \ldots,\ \sqrt[q-1]{x - s_n}\big)$$

and, for $i = 1, \ldots, n$,

$$L_i = \bar{\mathbb{F}}_q\big(\sqrt[q-1]{x - s_1},\ \sqrt[q-1]{x - s_2}, \ldots,\ \sqrt[q-1]{x - s_{i-1}},\ \sqrt[q-1]{x - s_{i+1}}, \ldots,\ \sqrt[q-1]{x - s_n}\big).$$

Denote $G_j = \mathrm{Gal}(L/L_j)$, for $j = \infty, 1, \ldots, n$. Then

$$G = G_\infty \times G_1 \times G_2 \times \cdots \times G_n \cong \widehat{\mathbb{Z}} \times \big(\mathbb{Z}/(q-1)\mathbb{Z}\big)^n.$$

Beside the Galois group G, the following are also derived from the basic datum.

Definition 1.1. We define the set of data $\gamma, m, t, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{i,j}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n$ as follows. The element t is a fixed generator of the multiplicative group $\mathbb{F}_q^*$. For $i = 1, \ldots, n$, put $B_i = g_i - 1 \in I$ where $g_i \in G_i$ is such that

$$g_i\big(\sqrt[q-1]{x - s_i}\big) = t \cdot \sqrt[q-1]{x - s_i}.$$

Also, put $A = F - 1 \in I$, where $F \in G_\infty$ is the Frobenius element.

The residue classes $m, a_1, \ldots, a_n, k_{i,j}$ $(1 \leqslant i < j \leqslant n)$ in $\mathbb{Z}/(q-1)\mathbb{Z}$ are defined so that

$$t^m = -1, \qquad t^{a_i} = (-1)^d f(s_i), \qquad t^{k_{i,j}} = s_i - s_j.$$

Finally, let $\gamma = 1 + q + q^2 + \cdots + q^{d-1} \in \mathbb{Z}$.

We shall call the tuple $(d, \gamma, m, t, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{i,j}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n, G)$ a derived datum. It depends on the choice of t. For convenience, we define $k_{i,j} = k_{j,i} + m$ if $i > j$.

Definition 1.2. A tame family Υ is a map which, to each basic datum $\lambda = (q, f, s_1, \ldots, s_n)$, assigns an element Υ_λ in the group ring $\mathbb{Z}[G]$ of the Galois group G. We will call Υ_λ a member of Υ.

Thus, in our setting, the family of Stickelberger elements forms a tame family. Roughly speaking, our main theorem says that there exists a formula to express every element in this family in terms of the derived datum and the formula remains the same for all the base field considered. For convenience, we first introduce some terminology.

Definition 1.3. We call an element $\mathcal{F} \in \mathbb{Z}[\mathbf{d}, \boldsymbol{\Gamma}, \mathbf{m}, \mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{k}_{1,2}, \ldots, \mathbf{k}_{n-1,n}, \mathbf{A}, \mathbf{B}_1, \ldots, \mathbf{B}_n]$ a universal polynomial of weight r and index N, where $r \in \{1, 2, \ldots\}$, $N \in \{0, 1, \ldots\} \cup \{-\infty\}$, if $\mathcal{F}$ is homogeneous of total degree r in the variables $\mathbf{A}, \mathbf{B}_1, \ldots, \mathbf{B}_n$ and the coefficient of $\mathbf{A}^r$ is $(-1)^r \gamma^N$ which is considered as zero if $N = -\infty$.

The polynomial $\mathcal{F}$ is called special if it does not involve the variables $\boldsymbol{\Gamma}$ and $\mathbf{m}$.

Then we make the following key definition.

Definition 1.4. A tame family Υ is said to be represented by a universal polynomial $\mathcal{F}$ if for each given basic datum $\lambda = (q, f, s_1, \ldots, s_n)$ with a derived datum $(d, \gamma, m, t, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n, G)$ the corresponding member Υ_λ is congruent to $\mathcal{F}(d, \gamma, m, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n)$ modulo I_G^{r+1}.

Remark. We should remind the reader that for a given basic datum λ, there are different choices of the generator t and hence different derived data. However, the definition requires that if $(d, \gamma, m, a'_1, \ldots, a'_n, k'_{1,2}, \ldots, k'_{n-1,n}, A', B'_1, \ldots, B'_n, G)$ is any other derived datum, we should have

$$\begin{aligned}&\mathcal{F}\big(d, \gamma, m, t', a'_1, \ldots, a'_n, k'_{1,2}, \ldots, k'_{n-1,n}, A', B'_1, \ldots, B'_n\big)\\ &\quad\equiv \mathcal{F}(d, \gamma, m, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n) \ \big(\mathrm{mod}\ I_G^{r+1}\big).\end{aligned}$$

Also, we want to point out that in the group ring, we have the obvious congruence

$$(g_1 - 1) + (g_2 - 1) \equiv g_1 g_2 - 1 \ \big(\mathrm{mod}\ I_G^2\big), \quad \text{for } g_1, g_2 \in G. \tag{2}$$

Therefore, if $g \in G$ is of finite order, then $\mathrm{ord}(g) \mid (q-1)$, and so

$$(q-1)(g-1) \equiv 0 \ \big(\mathrm{mod}\ I_G^2\big). \tag{3}$$

From this we see that although $m, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{n-1,n}$ are only defined modulo $q-1$, the residue class of $\mathcal{F}(d, \gamma, m, a_1, \ldots, a_n, k_{1,2}, \ldots, k_{n-1,n}, A, B_1, \ldots, B_n)$ modulo I_G^{r+1} is well-defined.

Our main result is the following. The special cases where $n = 2, 3$ are already proved in [8,14].

Theorem 1.5. *The tame family Θ of Stickelberger elements is represented by a universal polynomial $\mathcal{F}_\theta$ of weight n, index* 1.

In Section 3, we will calculate several related objects and complete the proof of the theorem in Section 3.4. In Section 2, we will show that the theorem together with a technique of shifting the basic datum can be used to prove weaker versions of the conjecture of Gross (Theorems 2.2, 2.3).

This is part of my PhD thesis, I would like to thank my advisor Ki-Seng Tan for many helpful suggestions and discussions from which many ideas in this paper are generated.

2. The conjecture of Gross

2.1. The regulator of Gross

The conjecture of Gross is a refinement of the class number formula. We first recall the definition of the regulator defined by Gross [6]. For the time being, we consider a more general situation where K is any global function field, S is a set of cardinality $n+1$, T is a non-empty set disjoint from S, and L/K is any profinite abelian extension unramified outside S. Denote by $\mathcal{O}_S$ the ring of S-integers in K. Let $U_{S,T}$ be the subgroup of units which are congruent to 1 modulo T and let $\mathrm{Pic}(\mathcal{O}_S)_T$ denote the group of invertible $\mathcal{O}_S$-modules together with a trivialization at T. Then $U_{S,T}$ is a free $\mathbb{Z}$-module and we have an exact sequence [6]

$$1 \to U_{S,T} \to \mathcal{O}_S^* \to \prod_{v\in T} \mathbb{F}_v^* \to \mathrm{Pic}(\mathcal{O}_S)_T \to \mathrm{Pic}(\mathcal{O}_S) \to 1. \tag{4}$$

In our case, $\mathrm{Pic}(\mathcal{O}_S)$ is trivial and hence

$$h_{S,T} := \big|\mathrm{Pic}(\mathcal{O}_S)_T\big| = \left|\mathrm{cokernel}\left(\mathcal{O}_S^* \to \prod_{v\in T} \mathbb{F}_v^*\right)\right|. \tag{5}$$

Let $\epsilon_1, \dots, \epsilon_n$ be a $\mathbb{Z}$-basis of $U_{S,T}$. For $v_1, \dots, v_n \in S$, let

$$r_i : K_{v_i}^* \to \mathrm{Gal}(L_{v_i}/K_{v_i}) \subset G$$

be the local reciprocity map. Then the determinant $\det(r_i(\epsilon_j) - 1)$ is in I^n, and the refined regulator $\mathcal{R}_{S,T}$ is defined as its residue class modulo I^{n+1}. The conjecture of Gross says [6]

$$\theta_{S,T} \equiv \pm h_{S,T} \cdot \mathcal{R}_{S,T} \pmod{I^{n+1}}. \tag{6}$$

Here the sign is determined by the orientation of the basis chosen and is consistent with the sign in the corresponding class number formula. For recent results on this conjecture and related subjects, see [1–4,6–8,10–12,15,17,18].

Now we return to our setting. It is difficult to find a basis of $U_{S,T}$. Instead of doing so, we follow [14] and consider the sub-module V spanned by

$$u_i := (x - s_i)^\gamma / \big((-1)^d f(s_i)\big), \quad i = 1, 2, \dots, n.$$

The index $(\mathcal{O}_S^* : V) = (q-1)\gamma^n$, and $(\mathcal{O}_S^* : U_{S,T}) = (q^d - 1)/h_{S,T}$ (see [14]). Therefore $(U_{S,T} : V) = \gamma^{n-1} h_{S,T}$, and

$$\gamma^{n-1} h_{S,T} \cdot \mathcal{R}_{S,T} \equiv \det\big(r_i(u_j) - 1\big) \pmod{I^{n+1}}. \tag{7}$$

Similar to [14], Proposition 3.7, we have

$$\begin{aligned} r_i(u_i) - 1 &\equiv -\gamma A + (dm - a_i)B_i - d\sum_{j\neq i} k_{i,j} B_j \pmod{I^2}, \\ r_i(u_j) - 1 &\equiv (dk_{i,j} - a_j)B_i \pmod{I^2}. \end{aligned} \tag{8}$$

To each basic datum we assign the element $\gamma^{n-1}h_{S,T}\cdot\mathcal{R}_{S,T}$ and this forms a tame family which we denote by $\mathfrak{V}$. By the above congruences (7) and (8), the family $\mathfrak{V}$ is represented by a universal polynomial which we denote by $\mathcal{F}_R$. In summary, we have the following lemma.

Lemma 2.1. *The tame family $\mathfrak{V}$ is represented by a universal polynomial $\mathcal{F}_R$ of weight n, index n.*

In next section, using Lemma 2.1 and Theorem 1.5, we will prove the following theorem.

Theorem 2.2. *Let L, S, T be as above. Then*

$$2\gamma^{n-1}\theta_{S,T}\equiv\pm 2\gamma^{n-1}h_{S,T}\mathcal{R}_{S,T}\ \left(\mathrm{mod}\ I^{n+1}\right).$$

Using this theorem together with the result of [15] (for the p-part) and the functorial argument used in [8,14], we can directly deduce the following theorem which is a generalization of the main theorems in [8,14].

Theorem 2.3. *Suppose $K=\mathbb{F}_q(x)$, $q\not\equiv 1\ (\mathrm{mod}\ 4)$, S is a set formed by $n+1$ degree-one places of K, T is a set formed by finitely many places such that the greatest common divisor of their degrees is relatively prime to $q-1$ and L/K is the maximal abelian extension unramified outside S. Then the conjecture of Gross holds.*

2.2. The shifting of the basic datum

Lemma 2.4. *Let $\mu>2$ be a given prime number. Suppose $\alpha_1,\dots,\alpha_l$ are elements of $\mathbb{Z}/\mu\mathbb{Z}$ and $\delta_1,\dots,\delta_l$ are integers such that their residue classes in the $\mathbb{F}_\mu$-vector space $\mathbb{Q}^*/(\mathbb{Q}^*)^\mu$ are linearly independent. Then there exist a prime p and a generator t of the multiplicative group $\mathbb{F}_p^*$ such that*

(1) $p\equiv 1\ (\mathrm{mod}\ \mu)$, *but* $p\not\equiv 1\ (\mathrm{mod}\ \mu^2)$.
(2) *If* $\delta_i\equiv t^{\beta_i}\ (\mathrm{mod}\ p)$ *for some integer* β_i, *then* $\beta_i\equiv\alpha_i\ (\mathrm{mod}\ \mu)$.

Proof. Let $K_1=\mathbb{Q}(\zeta_\mu)$ and $K_2=\mathbb{Q}(\zeta_{\mu^2},b_1,\dots,b_l)$, where ζ_N denotes a primitive Nth root of 1 and $b_i^\mu=\delta_i$. The given conditions imply $\mathrm{Gal}(K_2/K_1)=(\mathbb{Z}/\mu\mathbb{Z})^{l+1}$, and, under this identification, for an element $\sigma=(\sigma_0,\sigma_1,\dots,\sigma_l)\in(\mathbb{Z}/\mu\mathbb{Z})^{l+1}$, we have $\sigma(\zeta_{\mu^2})=\zeta_\mu^{\sigma_0}\zeta_{\mu^2}$ and $\sigma(b_i)=\zeta_\mu^{\sigma_i}b_i$.

Choose a σ such that $\sigma_0\neq 0$ and $\sigma_i=\alpha_i$ for $i=1,\dots,l$, and view it as an element in $\mathrm{Gal}(K_2/\mathbb{Q})$. By Tchebotarev's density theorem (see [9, p. 169]), there is a finite place w of K_2 such that at w, the numbers $b_1,\dots,b_l$ are local units and the corresponding Frobenius element equals to σ. Let p be the rational prime lying below w. Since σ fixes K_1, p splits completely in K_1, and hence $p\equiv 1\ (\mathrm{mod}\ \mu)$. Also, since $\sigma_0\neq 0$, p does not split under the extension K_2/K_1, and we must have $p\not\equiv 1\ (\mathrm{mod}\ \mu^2)$. This proves (1).

For an algebraic integer $\eta\in K_2$, denote its residue class modulo w by $\bar\eta$. Since σ acts as Frobenius, we have

$$\bar b_i^p=\bar\sigma(b_i)=\bar\zeta_\mu^{\alpha_i}\bar b_i,$$

and consequently,

$$\bar b_i^{p-1}=\bar\zeta_\mu^{\alpha_i}.$$

Now $p-1=\mu\varrho$, with $\mu\nmid\varrho$. Choose a generator t of $\mathbb{F}_p^*$ such that $\bar\zeta_\mu=t^\varrho$. If $\bar\delta_i=t^{\beta_i}$, then

$$t^{\beta_i\varrho}=\bar\delta_i^{\varrho}=\bar b_i^{\mu\varrho}=\bar b_i^{p-1}=t^{\alpha_i\varrho}.$$

Since ϱ is prime to μ, (2) is proved. □

Before proving Theorem 2.2, we consider the following.

Lemma 2.5. *If $v \notin S$ is the zero of a monic irreducible polynomial $h(x)$ of degree d', then its Frobenius element* $\mathrm{Fr} \in G$ *satisfies* $\mathrm{Fr}|_{G_\infty} = F^{d'}$ *and* $\mathrm{Fr}(\sqrt[q-1]{x-s_i}) = (-1)^{d'} h(s_i) \cdot \sqrt[q-1]{x-s_i}$ *for* $i = 1, \dots, n$.

Proof. This is directly from Class Field Theory. See [14, Lemma 3.4], for details. □

Lemma 2.6. *Suppose μ is a prime number. For every positive integer ν, there are distinct integers $\acute{s}_1, \dots, \acute{s}_\nu$ such that under the natural projection, the image of $\{\acute{s}_i - \acute{s}_j \mid 1 \leqslant i < j \leqslant \nu\}$ is linearly independent in the $\mathbb{F}_\mu$-linear space $\mathbb{Q}^*/(\mathbb{Q}^*)^\mu$.*

Proof. For $\nu = 2$ this is trivial. We prove the lemma by induction on ν. Suppose $\acute{s}_1, \dots, \acute{s}_\nu$ are chosen with the required property. We choose primes $p_1, p_2, \dots, p_\nu$ that does not divide any $\acute{s}_i - \acute{s}_j$. By the Chinese Remainder Theorem, there exists number x such that $p_i \mid x - s_i$ but $p_i^2 \nmid x - s_i$ and $p_i \nmid x - s_j$ if $i \neq j$. Then we put $\acute{s}_{\nu+1} = x$. □

Proof of Theorem 2.2. Lemma 2.1 and Theorem 1.5 together imply that the tame family Υ_0 formed by $2\gamma^{n-1}\theta_{S,T} - 2\gamma^{n-1}h_{S,T}\mathcal{R}_{S,T}$ is represented by the universal polynomial $\mathcal{Z} := 2\boldsymbol{\Gamma}^{n-1}\mathcal{F}_\theta - 2\mathcal{F}_R$ of weight n, index $-\infty$. Since $\gamma - d$ and $2m$ are all divisible by $q-1$ and $\mathcal{Z}$ is of index $-\infty$, the congruence (3) says that if in the expression of $\mathcal{Z}$ we discard the monomials involving $\mathbf{m}$ and change the variable $\boldsymbol{\Gamma}$ into the variable $\mathbf{d}$, then we get a special polynomial (of weight n and index $-\infty$)

$$\mathcal{Y} = \mathcal{Z}(\mathbf{d}, \mathbf{d}, 0, \mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{k}_{1,2}, \dots, \mathbf{k}_{n-1,n}, \mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_n)$$

which is in $\mathbb{Z}[\mathbf{d}, \mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{k}_{1,2}, \dots, \mathbf{k}_{n-1,n}, \mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_n]$ and also represents the tame family Υ_0.

Fix a given basic datum $(q, f, s_1, \dots, s_n)$ and for every i and (j,l) lift the derived $a_i, k_{j,l}$ to integers $\tilde{a}_i, \tilde{k}_{j,l}$. Put

$$\mathcal{Y}_0 = \mathcal{Y}(d, \tilde{a}_1, \dots, \tilde{a}_n, \tilde{k}_{1,2}, \dots, \tilde{k}_{n-1,n}, \mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_n).$$

It is enough to show $\mathcal{Y}_0 = 0$. We do it by showing that for any odd prime number μ, the residue class $\mathcal{Y}_{(\mu)}$ of $\mathcal{Y}_0$ modulo μ is zero.

Choose distinct integers $\acute{s}_1, \dots, \acute{s}_n$ satisfying the condition of Lemma 2.6. Then we use Lemma 2.4 to choose a prime number p and a generator t' of $\mathbb{F}_p^*$ such that if the $k'_{i,j}$th power of t' is congruent to $\acute{s}_i - \acute{s}_j$ modulo p, then

$$k'_{i,j} \equiv \tilde{k}_{i,j} \pmod{\mu}.$$

For each i, let s'_i be the residue class of $\acute{s}_i$ modulo p. From Lemma 2.5 and Tchebotarev's density theorem, we can find an irreducible polynomial $f' \in \mathbb{F}_p[x]$ with $d' := \deg(f')$ congruent to d modulo μ and $f'(s'_i) = (-1)^{d'} t'^{\tilde{a}_i}$ for every i. Thus we have constructed a basic datum $(p, f', s'_1, \dots, s'_n)$ with a derived datum $(d', \gamma', m', t', a'_1, \dots, a'_n, k'_{1,2}, \dots, k'_{n-1,n}, A', B'_1, \dots, G')$ such that every a'_i is congruent to $\tilde{a}_i$ and every $k'_{i,j}$ is congruent to $\tilde{k}_{i,j}$ modulo μ.

Let $G'_{(\mu)} \cong (\mathbb{Z}/\mu\mathbb{Z})^{n+1}$ be the maximal elementary μ-quotient of G' and for an element $\xi \in \mathbb{Z}[G']$ denote by $\xi_{(\mu)} \in \mathbb{Z}[G'_{(\mu)}]$ its image under the natural projection. Then $(2\gamma^{n-1}\theta_{S',T'} - 2\gamma^{n-1}h_{S',T'}\mathcal{R}_{S',T'})_{(\mu)}$ and $\mathcal{Y}_{(\mu)}((A')_{(\mu)}, (B'_1)_{(\mu)}, \dots, (B'_n)_{(\mu)})$ are congruent modulo $I_{G'_\mu}^{n+1}$. But the conjecture of Gross holds for elementary groups ([10], since K has class number 1), and this implies $\mathcal{Y}_{(\mu)} = 0$ ([13] or [16, Proposition 3.8]). □

3. Universal polynomials

3.1. Some combinatorial formulae

We recall that the combinatorial symbol $\binom{\alpha}{\beta}$ is defined for $\alpha, \beta \in \mathbb{Z}$ and $\binom{\alpha}{\beta} = 0$ if either $\beta < 0$ or $0 < \alpha < \beta$. The following three well-known lemmas will be useful for us.

Lemma 3.1. *Suppose ξ, ψ are non-negative integers and k is a positive integer. Then for a sequence of non-negative integers $\beta_1, \ldots, \beta_k$ such that $\sum_{i=1}^{k} \beta_i = \psi$, we have*

$$\sum \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_k}{\beta_k} = \binom{\xi + k - 1}{\psi + k - 1},$$

where the sum is over all k-tuples of non-negative integers $(\alpha_1, \ldots, \alpha_k)$ that satisfy $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \xi$.

Proof. The case $k = 2$ is from [5, p. 169, (5.26)], and we can prove it by induction on k. We have

$$\begin{aligned}\sum \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_k}{\beta_k} &= \sum_{\alpha_1} \binom{\alpha_1}{\beta_1} \sum_{\alpha_2 + \cdots + \alpha_k = \xi - \alpha_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_k}{\beta_k} \\ &= \sum_{\alpha_1} \binom{\alpha_1}{\beta_1} \binom{\xi - \alpha_1 + k - 1 - 1}{\psi - \beta_1 + k - 1 - 1} \\ &= \binom{\xi + k - 1}{\psi + k - 1}. \qquad \square\end{aligned}$$

The following two lemmas are from [5, p. 169, (5.25)], with the observation that $\binom{-\alpha}{\beta} = (-1)^{\beta}\binom{\alpha+\beta-1}{\beta}$ for $\alpha, \beta \in \mathbb{N}$.

Lemma 3.2. *For non-negative integers ξ, ψ and k with $\xi \geqslant \psi > k$, we have*

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{\xi - i}{\psi} = \binom{\xi - k}{\psi - k}.$$

Lemma 3.3. *For non-negative integers ξ, ψ, k with $\xi \geqslant \psi > k$, we have*

$$\sum_{i=0}^{\psi - k} (-1)^i \binom{\xi}{i} \binom{\psi - i}{k} = (-1)^{\psi - k} \binom{\xi - k - 1}{\psi - k}.$$

3.2. The variance

From now on, let H denote the subgroup $G_1 \times \cdots \times G_n \subset G$ and put $q^* = q - 1$. Recall that $m = q^*/2$ if q^* is even, and $m = 0$ if q^* is odd.

Definition 3.4. Define the variance $\Sigma(H)$ as the sum $\sum_{\sigma \in H} (\sigma - 1) \in \mathbb{Z}[H]$. Also, for every integer r between 1 and n, denote $H^{(r)} = G_1 \times \cdots \times G_r$ and $H_{(r)} = G_{r+1} \times \cdots \times G_n$, and put $\Sigma^{(r)} = \sum_{\sigma \in H^{(r)}} (\sigma - 1)$ and $\Sigma_{(r)} = \sum_{\sigma \in H_{(r)}} (\sigma - 1)$.

The following lemma is from [14].

Lemma 3.5. *Suppose $\sigma \in H$. If q^* is odd, then $q^*(\sigma - 1) \equiv 0 \pmod{I^3}$. If q^* is even, then $q^*(\sigma - 1) \equiv m(\sigma - 1)^2 \pmod{I^3}$.*

Suppose J is a non-empty subset of $\{1, \dots, r\}$ and $\{\alpha_j \mid j \in J\}$ is a set of natural numbers such that $\sum_{j\in J} \alpha_j = r$. By Lemma 3.5, the residue class modulo $I_{H^{(r)}}^{r+1}$ of $F_J^{(r)} := m^r \prod_{j\in J} B_j^{\alpha_j}$ is independent of the choice of α_j's. Let Δ_r the set of all non-empty subsets of $\{1, \dots, r\}$.

Lemma 3.6. *We have*

$$\Sigma^{(r)} \equiv \sum_{J\in\Delta_r} F_J^{(r)} \left(\operatorname{mod} I_{H^{(r)}}^{r+1}\right).$$

Proof. In this proof, we will forget our setting and just view the lemma as a statement about the group $(\mathbb{Z}/q^*\mathbb{Z})^n$. Thus, we can let n vary, and only need to treat the case $n = r$. We will prove it by induction on n. By (2), we have

$$\Sigma^{(1)} = \sum_{i=0}^{q^*-1} \left(\sigma_1^i - 1\right) \equiv \sigma_1^{q^*(q^*-1)/2} - 1 \equiv \frac{q^*(q^*-1)}{2} B_1 \equiv mB_1 \equiv \sum_{J\in\Delta_1} F_J^{(1)} \left(\operatorname{mod} I_{H^{(1)}}^2\right).$$

This proves the case $n = 1$.

Now $H^{(n)} = H^{(n-1)} \times H_{(n-1)}$. From the equality

$$\sigma\rho - 1 = (\sigma - 1)(\rho - 1) + \sigma - 1 + \rho - 1$$

we see that

$$\Sigma^{(n)} = \Sigma^{(n-1)} \Sigma_{(n-1)} + q^* \Sigma^{(n-1)} + (q^*)^{n-1} \Sigma_{(n-1)}.$$

The induction hypothesis and Lemma 3.5 imply that $(q^*)^{n-1} \Sigma_{(n-1)} = F_{\{n\}}^{(n)}$ and $q^* \Sigma^{(n-1)} = \sum_{J\in\Delta_{n-1}} F_J^{(n)}$. Also, if Δ_n' be the subset of Δ_n consisting of J such that $n \in J$ and $|J| > 1$, then $\Sigma^{(n-1)} \Sigma_{(n-1)} = \sum_{J\in\Delta_n'} F_J^{(n)}$. We complete the proof by noticing that

$$\Delta_n = \Delta_n' \cup \Delta_{n-1} \cup \{\{n\}\}. \qquad \square$$

3.3. Representable families

From now on, under the natural embedding $H^{(r)} \hookrightarrow H$, we will identify $\mathbb{Z}[H^{(r)}]$ as a subring of $\mathbb{Z}[H]$. In this section we will consider several tame families representable by universal polynomials. To simplify the notations, we will use a general form of the members to denote the family. For instance, we will say that $\Sigma(H)$ is a tame family and so is $\Sigma^{(r)}$ for any $r = 1, \dots, n$.

Definition 3.7. We denote by Ω_r the set of tame families representable by a universal polynomial of weight r and index $-\infty$.

Thus the tame family $\Sigma^{(r)}$ is in Ω_r (Lemma 3.6).

Definition 3.8. For a non-negative integer k and $l = 1, 2, \ldots, n$, define $\Lambda_k^{(l)}$ to be the set of all monic degree k polynomials $\zeta(x) \in \mathbb{F}_q[x]$ such that $\zeta(s_i) \neq 0$ for every $i = 1, \ldots, l$. Define $\Lambda^{(l)} = \bigcup_{i=0}^{\infty} \Lambda_i^{(l)}$. Also, define $\phi^{(l)} : \Lambda^{(l)} \to H^{(l)}$ such that for $\zeta \in \Lambda_k^{(l)}$, $\phi^{(l)}(\zeta) = (h_1, \ldots, h_l)$ with $h_i(\sqrt[q-1]{x - s_i}) = (-1)^k \zeta(s_i) \cdot \sqrt[q-1]{x - s_i}$.

Note that we have

$$\left|\Lambda_k^{(l)}\right| = \sum_{i=0}^{k} (-1)^i \binom{l}{i} q^{k-i}. \tag{9}$$

Lemma 3.9. *Put*

$$W = \left(\phi^{(n)}(f) - 1\right)\left(\sum_{k=0}^{n-1} \left|\Lambda_k^{(n)}\right| F^k\right).$$

We have $W \in \Omega_n$.

Proof. By Lemma 2.5, we see that $\phi^{(n)}(f) - 1$ is congruent to $\sum_{i=1}^{n} a_i B_i$ modulo I^2 and therefore it is in Ω_1. By Lemma 3.5, we then have $(\phi^{(n)}(f) - 1)(F - 1)^r \in \Omega_{r+1}$ and $(q^*)^r(\phi^{(n)}(f) - 1) \in \Omega_{r+1}$. The lemma is proved by these and

$$\begin{aligned}
\sum_{k=0}^{n-1} \left|\Lambda_k^{(n)}\right| F^k &= \sum_{k=0}^{n-1} \sum_{j=0}^{k} (-1)^j \binom{n}{j} q^{k-j} F^k \\
&= \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} \sum_{k=j}^{n-1} q^{k-j} F^k \\
&= \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} \frac{(qF)^{n-j} - 1}{qF - 1} \cdot F^j \\
&= \frac{1}{qF - 1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} F^j (qF)^{n-j} - F^j \\
&= \frac{1}{qF - 1}\left((qF - F)^n - (1 - F)^n\right) \\
&= \sum_{i=0}^{n-1} (q^* F)^i (1 - F)^{n-1-i}. \qquad \square
\end{aligned}$$

Definition 3.10. For a non-negative integer k define $U_k = \sum(\phi^{(n)}(\zeta) - 1) \in \mathbb{Z}[H]$ where ζ runs through the set $\Lambda_k^{(n)}$. Also, define $X_k = \sum_{i=0}^{k} \binom{n-k+i-1}{i} U_{k-i}$ and $Y_k = \sum_{i=n-1-k}^{n-1} \binom{i}{n-1-k} U_i$. For $l = 1, \ldots, n$ denote by $\bar{U}_k^{(l)}$, $\bar{X}_k^{(l)}$, $\bar{Y}_k^{(l)}$ the image of U_k, X_k, Y_k under the natural projection $\mathbb{Z}[H] \to \mathbb{Z}[H^{(l)}]$.

In particular, we have $X_0 = U_0 = 0$ and

$$X_1 = U_1 = \sum_{s \in \mathbb{F}_q \setminus \{s_1, \ldots, s_n\}} \left(\phi^{(n)}(x - s) - 1\right).$$

Using (2) and the equality $\prod_{s\in\mathbb{F}_q\setminus\{s_1,\dots,s_n\}}(s_i-s)=-\prod_{j\neq i}(s_i-s_j)^{-1}$, we get

$$X_1\equiv\sum_{i=1}^{n}\Bigg(m-\sum_{j\neq i}k_{i,j}\Bigg)B_i\ \big(\mathrm{mod}\ I^2\big),$$

and consequently,

$$X_1\in\Omega_1. \tag{10}$$

Lemma 3.11. *For $k=1,\dots,n-1$, we have $X_k\in\Omega_k$.*

Proof. We prove it by induction on k. The case where $k=1$ is done above. For a non-negative integer i, let Ψ_i be the set consisting of ith degree monomials of the form $(x-s_{k+1})^{\alpha_{k+1}}\cdots(x-s_n)^{\alpha_n}$ with $\alpha_j\geqslant 0$. Then every element in $\Lambda_k^{(k)}$ can be expressed as a product $\eta\zeta$ with $\eta\in\Psi_i$, $\zeta\in\Lambda_{k-i}^{(n)}$, for some i. Consider the sum

$$Z_k:=\sum_{\zeta\in\Lambda_k^{(k)}}\phi^{(k)}(\zeta)-1=\sum_{i=0}^{k}\sum_{\eta\in\Psi_i}\sum_{\zeta\in\Lambda_{k-i}^{(n)}}\phi^{(k)}(\eta\zeta)-1. \tag{11}$$

We first note that the restriction of $\phi^{(k)}$ to $\Lambda_k^{(k)}$ is an injection into $H^{(k)}$ and since $|\Lambda_k^{(k)}|=(q^*)^k=|H^{(k)}|$ (see (9)), it is actually a bijection between these two sets. Consequently we have

$$Z_k=\Sigma^{(k)}\in\Omega_k. \tag{12}$$

We also have $|\Psi_i|=\binom{n-k+i-1}{i}$ and $\bar{X}_k^{(k)}=\sum_{i=0}^{k}(|\Psi_i|\sum_{\zeta\in\Lambda_{k-i}^{(n)}}(\phi^{(k)}(\zeta)-1))$. Compare the above two sums, we get

$$Z_k-\bar{X}_k^{(k)}=\sum_{i=1}^{k}\sum_{\eta\in\Psi_i}\Bigg(\big(\phi^{(k)}(\eta)-1\big)\sum_{\zeta\in\Lambda_{k-i}^{(n)}}\phi^{(k)}(\zeta)\Bigg). \tag{13}$$

For $i=k+1,\dots,n$, put $C_i=\phi^{(k)}(x-s_i)-1\in I_H$, and for each positive integer l, put

$$C_{[l]}=\sum_{k+1\leqslant j_1\leqslant\cdots\leqslant j_l\leqslant n}C_{j_1}\cdots C_{j_l}.$$

If $\eta=(x-s_{k+1})^{\alpha_{k+1}}\cdots(x-s_n)^{\alpha_n}$, then $\phi^{(k)}(\eta)-1=(C_{k+1}+1)^{\alpha_{k+1}}\cdots(C_n+1)^{\alpha_n}-1$ and Lemma 3.1 implies

$$\sum_{\eta\in\Psi_i}\big(\phi^{(k)}(\eta)-1\big)=\sum_{l=1}^{i}\binom{i+n-k-1}{l+n-k-1}C_{[l]}. \tag{14}$$

Also, we have $\sum_{\zeta\in\Lambda_{k-i}^{(n)}}\phi^{(k)}(\zeta)=\bar{U}_{k-i}^{(k)}+|\Lambda_{k-i}^{(n)}|$. This together with (13) and (14) implies

$$Z_k - \bar{X}_k^{(k)} = \sum_{l=1}^{k}\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}\bar{U}_{k-i}^{(k)}C_{[l]} + \sum_{l=1}^{k}\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}|\Lambda_{k-i}^{(n)}|C_{[l]}. \tag{15}$$

By (9), we have

$$\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}|\Lambda_{k-i}^{(n)}| = \sum_{j=l}^{k}\sum_{i=l}^{j}\binom{i+n-k-1}{l+n-k-1}(-1)^{j-i}\binom{n}{j-i}q^{k-j}.$$

Replace $j-i$ by i and use Lemma 3.3 ($\xi = n$, $\psi = j+n-k-1$, $k = l+n-k-1$), and we get

$$\begin{aligned}\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}|\Lambda_{k-i}^{(n)}| &= \sum_{j=l}^{k}\sum_{i=0}^{j-l}(-1)^i\binom{n}{i}\binom{j-i+n-k-1}{l+n-k-1}q^{k-j}\\ &= \sum_{j=l}^{k}(-1)^{j-l}\binom{k-l}{j-l}q^{k-j}\\ &= (q^*)^{k-l}.\end{aligned}$$

Now, by Lemma 3.5, $\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}|\Lambda_{k-i}^{(n)}|c_{[l]}$ is in Ω_k. By induction hypothesis,

$$\sum_{i=l}^{k}\binom{i+n-k-1}{l+n-k-1}\bar{U}_{k-i}^{(k)}C_{[l]} = \bar{X}_{k-l}C_{[l]} \in \Omega_k.$$

Therefore, from (12) and (15), we see that $\bar{X}_k^{(k)} \in \Omega_k$. The nth symmetric group acts on our settings through its action on $\{s_1, \dots, s_n\}$. From this, we see that under any homomorphism of the type $H \to G_{i_1} \times G_{i_2} \times \cdots \times G_{i_k} \hookrightarrow H$, where $1 \leqslant i_i < i_2 < \cdots < i_k \leqslant n$, the first arrow is the natural projection and the second is the natural embedding, the image of X_k is in Ω_k. Then this shows that $X_k \in \Omega_k$ [12, Lemma 2]. □

Corollary 3.12. *For $k = 0, 1, 2, \dots, n-1$, we have $Y_k \in \Omega_k$.*

Proof. We have, for $k = 0, 1, \dots, n-1$,

$$\sum_{i=k}^{n-1}(-1)^{n-1-i}\binom{i}{k}Y_i = \sum_{j=0}^{n-1}\sum_{i=n-1-j}^{n-1}(-1)^{n-1-i}\binom{i}{k}\binom{j}{n-1-i}U_j.$$

Replace i by $n-1-i$ and apply Lemma 3.2, we see the above equals to $\sum_{j=0}^{n-1}\binom{n-1-j}{k-j}U_j$ which is actually X_k. Thus $Y_{n-1} = X_{n-1}$ and Y_k can be expressed as a linear combination of X_k and $Y_{k+1}, \dots, Y_{n-1}$. □

3.4. Computing $\theta_{S,T}$

Lemma 3.13. *Let L, S, T be as in Theorem 2.2, we have*

$$\theta_{S,T} = \left(1 - q^d F^d \phi^{(n)}(f)\right)\left(\sum_{k=0}^{n-1}F^k \sum_{\zeta \in \Lambda_k^{(n)}}\phi^{(n)}(\zeta)\right) + F^n\left(\sum_{i=0}^{d-1}q^i F^i\right)\sum_{\sigma \in H}\sigma.$$

Proof. Similar to [8, Lemma 3.6]. Here we give a brief sketch of the proof. For a place $v \notin S$, let $\mathrm{Fr}_v \in G$ denote the Frobenius element at v. Then $1 - \mathrm{Fr}_v N(v)^{-s}$ is invertible in the formal power series ring $\mathbb{Z}[G][\![q^{-s}]\!]$ over the group ring $\mathbb{Z}[G]$. Put

$$\mathbf{L}_{S,T} = \mathbf{L}_S(s) \prod_{v \in T} \bigl(1 - \mathrm{Fr}_v N(v)^{1-s}\bigr),$$

where

$$\mathbf{L}_S(s) = \prod_{v \notin S} \bigl(1 - \mathrm{Fr}_v N(v)\bigr)^{-1} = \sum_{\eta \in \Lambda^{(n)}} F^{\deg(\eta)} \phi^{(n)}(\eta) q^{-s\deg(\eta)}.$$

We can write $\mathbf{L}_S(s) = \sum_{i=0}^{\infty} M_i F^i q^{-is}$, where $M_i = \sum_{\eta \in \Lambda_i^{(n)}} \phi^{(n)}(\eta)$. As we have seen before, the restriction of the map $\phi^{(n)}$ gives a bijection from $\Lambda_n^{(n)}$ to H. Therefore, $M_n = \sum_{\sigma \in H} \sigma$, and similarly for $i \geqslant n$, we have $M_i = q^{i-n} \sum_{\sigma \in H} \sigma$.

Now, as T is formed by the zero of $f(x)$, we have

$$\mathbf{L}_{S,T}(s) = \bigl(1 - q^d F^d \phi^{(n)}(f) q^{-sd}\bigr) \sum_{i=0}^{\infty} M_i F^i q^{-is}.$$

The term $M_n F^n q^{-ns}$ multiplying with $q^d F^d \phi^{(n)}(f) q^{-ds}$ will cancel with the term $M_{d+n} F^{d+n} q^{-(d+n)s}$. Consequently, $\mathbf{L}_{S,T}$ is actually a polynomial in q^{-s} with degree at most $d+n$, and $\theta_{S,T} = \mathbf{L}_{S,T}(0)$ is of the desired form. □

Proof of Theorem 1.5. Consider the following decompositions

$$\sum_{\zeta \in \Lambda_k^{(n)}} \phi^{(n)}(\zeta) = U_k + \bigl|\Lambda_k^{(n)}\bigr|, \tag{16}$$

$$\sum_{\sigma \in H} \sigma = \Sigma^{(n)} + (q^*)^n, \tag{17}$$

$$1 - q^d F^d \phi^{(n)}(f) = \bigl(1 - q^d F^d\bigr) - q^d F^d \bigl(\phi^{(n)}(f) - 1\bigr). \tag{18}$$

By Lemma 3.13, (16), (17) and (18), we have

$$\theta_{S,T} = \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV},$$

where

$$\mathbf{I} = \bigl(1 - q^d F^d \phi^{(n)}(f)\bigr) \sum_{k=0}^{n-1} F^k U_k,$$

$$\mathbf{II} = q^d F^d W \quad \text{(Lemma 3.9)},$$

$$\mathbf{III} = \bigl(1 - q^d F^d\bigr) \sum_{k=0}^{n-1} F^k \bigl|\Lambda_k^{(n)}\bigr| + (q^*)^n F^n \left(\sum_{i=0}^{d-1} q^i F^i \right),$$

$$\mathbf{IV} = F^n \left(\sum_{i=0}^{d-1} q^i F^i \right) \Sigma^{(n)}.$$

Using the equality $F = A + 1$, we get

$$1 - q^d F^d \phi^{(n)}(f) = \left(1 - q^d\right) + q^d \sum_{i=1}^{d} \binom{d}{i} A^i \phi^{(n)}(f),$$

$$\sum_{k=0}^{n-1} F^k U_k = \sum_{j=1}^{n-1} \sum_{k=j}^{n-1} \binom{k}{j} U_k A^j.$$

The last sum is just $\sum_{j=1}^{n-1} Y_{n-1-j} A^j$. Lemma 2.5 implies that $\phi^{(n)}(f) - 1$ is congruent to $\sum_{i=1}^{n} a_i B_i$ modulo I^2. Since $qB_i \equiv B_i \pmod{I^2}$, by Corollary 3.12, we have $\mathbf{I} \in \Omega_n$. Also, Lemma 3.6 and Lemma 3.9 say both W and $\Sigma^{(n)}$ are in Ω_n, and $\mathbf{II} \equiv W \pmod{I^{n+1}}$, $\mathbf{IV} \equiv d\Sigma^{(n)} \pmod{I^{n+1}}$. Finally, we have

$$\begin{aligned}
\mathbf{III} &= \sum_{i=0}^{d-1} q^i F^i \left((1 - qF) \sum_{k=0}^{n-1} F^k \left| \Lambda_k^{(n)} \right| + (q^*)^n F^n \right) \\
&= \sum_{i=0}^{d-1} q^i F^i \left(1 + \left(\sum_{k=1}^{n-1} F^k \left(\left| \Lambda_k^{(n)} \right| - q \left| \Lambda_{k-1}^{(n)} \right| \right) \right) - qF^n \left| \Lambda_{n-1}^{(n)} \right| + (q^*)^n F^n \right) \\
&= \sum_{i=0}^{d-1} q^i F^i \left(1 + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} F^k + (-1)^n F^n \right) \quad \left(\text{by Eq. (9)}\right) \\
&= (1 - F)^n \sum_{i=0}^{d-1} q^i F^i \\
&\equiv (-1)^n \gamma A^n \left(\text{mod } I^{n+1}\right). \qquad \square
\end{aligned}$$

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